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Block-Iterative Subgradient Projection Algorithms for the Convex Feasibility Problem*

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Abstract In this paper, sequential block-iterative subgradient projection algorithm and parallel block-iterative subgradient projection algorithm for solving the convex feasibility problem expressed by the system of inequalities are presented. Each step in these methods consists of finding the approximation projection of the current point on the subsystem which is constructed through parting the system of inequalities into several blocks. The convergence for both of sequential block-iterative subgradient projection algorithm and parallel block-iterative subgradient projection algorithm are obtained under some weak conditions.

Keywords Operations research, convex function, convex feasibility problem, subgradient, convergence

Subject Classification (GB/T13745-92) 110.74

凸可行问题的块迭代次梯度投影算法

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摘要 本文，针对由非线性不等式系统构成的凸可行问题，提出了序列块迭代次梯度投影算法和平行块迭代次梯度投影算法．将非线性不等式系统分成若干个子系统，然后将当前迭代点在子系统各个子集上的次梯度投影的凸组合作为当前迭代点在这个子系统上的近似投影．在较弱条件下证明了两种算法的收敛性．

关键词 运筹学，凸函数，凸可行问题，次梯度，收敛性

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1 Introduction

The convex feasibility problem(CFP) is to find a point x in the intersection C of m closed convex sets $C_1, \cdots, C_m \subset \Re^n$. Without loss of generality we may assume that $C_i, i = 1, 2, \cdots, m$ are expressed as

$$C_i = \{x \in \Re^n : f_i(x) \leqslant 0\},$$

where $f_i : \Re^n \to \Re, i = 1, 2, \cdots, m$ are convex. In this formulation, the CFP consists of finding a feasible solution to the system of inequalities

$$f_i(x) \leqslant 0, \quad 1 \leqslant i \leqslant m.$$

It is well known that the CFP has many applications in some diverse areas of mathematics and engineering, for instance optimization[1−2], approximation theory[3], image reconstruction from projections and computerized tomography[4−5], control theory[6]. A popular approach to such problem is projection algorithm, see [7]. The so called "relaxation methods" for solving the convex feasibility problem which date back to Kaczmarz[8] and Cimmion[9] are of special interest by reason of their relatively easy implementation and computational efficiency in solving extremely large and sparse problems. Classical contributation to the study of relaxation methods are surveyed in [5]. Aharoni and Censor discussed a "block-iterative projection method" which incorporates as special case many of the earlier relaxation techniques (see [10] and the references therein). But a serious obstacle for the successful implementation of these algorithms lies in the computation of the projections since no closed formulas like linear system is available, while subgradient projection algorithms can overcome the obstacle. There came out cyclic subgradient projections (CSP)[2], parallel subgradient projections (PSP)[11], Eremin's algorithmic scheme[12], and the others see [13-15]. In this paper, we present a sequential block-iterative subgradient projection algorithm and a parallel block-iterative subgradient projection algorithm for solving the convex feasibility problem, which not only compute easily but also can improve the convergence.

The rest of the paper is organized as follows. In Section 2, we review the cyclic subgradient projection and the parallel subgradient projection. In Section 3, we present a sequential block-iterative subgradient projection algorithm and its convergence, and present a parallel block-iterative subgradient projection algorithm and its convergence. In Section 4, some conclusions are drawn.

2 The basic algorithms

Definition 2.1(subdifferential) Let $f : \Re^n \to \Re$ be convex. The subdifferential of f at x is defined as

$$\partial f(x) = \{\xi \in R^n |\ f(y) \geqslant f(x) + \langle \xi, y - x \rangle, \quad \forall\ y \in \Re^n\},$$

ξ is said to be a subgradient of f at the point x.

Let $I = \{1, 2, \cdots, m\}$. Below is the cyclic subgradient projection (CSP) method for the CFP presented by Censor and Lent[5].

The cyclic subgradient projections method(CSP)

Initialization: $x^0 \in \Re^n$ is arbitrary.

Iterative step: Given x^k, $i \in I$, compute the next iteration x^{k+1} from

$$x^{k+1} = \begin{cases} x^k - \lambda \dfrac{f_i(x^k)}{\|\xi_i^k\|^2} \xi_i^k, & \text{if } f_i(x^k) > 0; \\ x^k, & \text{if } f_i(x^k) \leqslant 0, \end{cases} \tag{2.1}$$

where $\xi_i^k \in \partial f_i(x^k)$ is a subgradient of f_i at the point x^k, and the relaxation parameters λ is confined to an interval $\varepsilon_1 \leqslant \lambda \leqslant 2 - \varepsilon_2$, for all $k \geqslant 0$, with some $\varepsilon_1, \varepsilon_2 > 0$.

The following is a method of parallel subgradient projections(PSP)[11] for solving the CFP.

The method of parallel subgradient projections(PSP)

Initialization: $x^0 \in \Re^n$ is arbitrary.

Iterative step: (i) Given x^k, for all $i \in I = \{1, 2, \cdots, m\}$, compute the intermediate iterative point $y^{k+1,i}$ by

$$y^{k+1,i} = \begin{cases} x^k - \lambda \dfrac{f_i(x^k)}{\|\xi_i^k\|^2} \xi_i^k, & \text{if } f_i(x^k) > 0; \\ x^k, & \text{if } f_i(x^k) \leqslant 0, \end{cases} \tag{2.2}$$

where $\xi_i^k \in \partial f_i(x^k)$ is a subgradient of f_i at the point x^k, and the relaxation parameters λ is confined to an interval $\varepsilon_1 \leqslant \lambda \leqslant 2 - \varepsilon_2$, for all $k \geqslant 0$, with some $\varepsilon_1, \varepsilon_2 > 0$.

(ii) Compute the next iteration x^{k+1} by

$$x^{k+1} = \sum_{i=1}^{m} \omega_i y^{k+1,i}, \tag{2.3}$$

where ω_i are fixed, user-chosen, positive weights with $\sum\limits_{i=1}^{m} \omega_i = 1$.

3 The block-iterative subgradient projections method and its convergence

Block-iterative algorithm works with groups of constraints, which may be obtained by parting the indices of I as $I = I_1 \bigcup I_2 \bigcup \cdots \bigcup I_p$ into p blocks. Usually, during our implementation, the blocks are divided almost evenly. Define $C_t^k = \{f_i(x^k), i \in I_t\}$ as the t-th subsystem, $f^+ = \max\{f, 0\}$.

3.1 The sequential block-iterative subgradient projections algorithm and its convergence

The main iteration of sequential block-iterative subgradient can proceed sequentially from block to block (according to some index control sequence, similar the cyclic subgradient projection between the blocks), while work is performed in parallel within each block. We present the corresponding subgradient projection method as follows.

Algorithm 3.1

Let x^k be the current point. We set $I_t^k = \{i : i\text{-th constraint in } t\text{-th subsystem is violated by } x^k\}$ and denote by $\omega_t^k = (\omega_{i(t)}^k : i \in I_t^k)$ the weight vector for the t-th subsystem, $t = 1, \cdots, p$ at the current point x^k, with $\sum_{i \in I_t^k} \omega_{i(t)}^k = 1$.

Let the t-th subsystem be the one to be operated next. Find I_t^k.

If $I_t^k = \emptyset$, define $x^{k+1} = x^k$, and if $t < p$ go to the next subsystem with x^{k+1}. If $t = p$, the major cycle is finished. If there is no change in the current point throughout this major cycle, then the current point is feasible to (1.1); terminate. Otherwise, go to the next major cycle with the current point.

If $I_t^k \neq \emptyset$, select a weight vector ω_t^k and define

$$x^{k+1} = x^k - \lambda\Big[\sum_{i\in I_t^k} \omega_i^k \frac{f_i(x^k)}{\|\xi_i^k\|^2}\xi_i^k\Big], \tag{3.1}$$

where $0 < \lambda < 2$. With x^{k+1}, go to the next subsystem if $t < p$, or to the next major cycle if $t = p$.

Next we discuss the convergence of the sequential block-iterative subgradient projection algorithm.

For the sake of simplicity, all our convergence proofs will be based on the assumption that the weight vector $\omega_t^k = (\omega_{i(t)}^k : i \in I_t^k)$ is selected so as to satisfy

$$\omega_{i(t)}^k > \gamma, i \in I_t^k, t = 1, 2, \cdots, p,$$

where γ is some predetermined small positive quantity. Let $x \in \Re^n$. Set

$$y^{k+1,i} = P_{i,\lambda}(x) = x - \lambda\frac{f_i^k}{\|\xi_i^k\|^2}\xi_i^k,$$
$$x^{k+1} = P_{\omega_t,\lambda}(x) = x - \lambda\sum_{i\in I_t}\omega_{i(t)}\frac{f_i^k}{\|\xi_i^k\|^2}\xi_i^k,$$

where $\lambda \in (0, 2)$.

Proposition 3.1 *If $x \in \Re^n$, then for every $y \in C_i$ and every $\lambda \in (0, 2)$,*

$$\|P_{i,\lambda}(x) - y\| \leqslant \|x - y\|. \tag{3.2}$$

Moreover, if $x \notin C_i$, then the inequality is strict.

Proof If $x \in C_i$ then the equality is set, let us assume that $x \notin C_i$. We have

$$\begin{aligned}\|P_{i,\lambda}(x) - y\|^2 &= \Big\|x - \lambda\frac{f_i^k}{\|\xi_i^k\|^2}\xi_i^k - y\Big\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle\frac{f_i(x^k)}{\|\xi_i^k\|^2}\xi_i^k, x - y\rangle + \lambda^2\frac{(f_i^k)^2}{\|\xi_i^k\|^2}.\end{aligned}$$

From the subgradient definition, we get

$$\begin{aligned}\|P_{i,\lambda}(x) - y\|^2 &\leqslant \|x - y\|^2 - 2\lambda\frac{(f_i(x^k))^2}{\|\xi_i^k\|^2} + \lambda^2\frac{(f_i(x^k))^2}{\|\xi_i^k\|^2} \\ &\leqslant \|x - y\|^2 - \lambda(2 - \lambda)\frac{(f_i(x^k))^2}{\|\xi_i^k\|^2} \\ &\leqslant \|x - y\|^2.\end{aligned} \tag{3.3}$$

Then, the proof of the proosition is completed.

Proposition 3.2 *Let $q \in C = \bigcap_{i\in I} C_i, \lambda \in (0,2)$, and let ω_t be weight vector. Then, for every $x \in \Re^n$,*

$$\|P_{\omega_t,\lambda}(x) - q\| \leqslant \|x - q\|. \tag{3.4}$$

Proof Obviously, $P_{\omega_t,\lambda}(x) = \sum\limits_{i\in I^t} \omega(i)P_{i,\lambda}(x), i \in I_t, t = 1,2,\cdots,p$, and repeat use of the proposition 1 with $y = q$ shows that $P_{i,\lambda}(x) \in B = B(q, \|x - q\|)$, where $B(x,\rho) = \{y \in \Re^n | \|x - y\| \leqslant \rho\}$ is the ball with radius ρ centered at $x \in \Re^n$. Then convex combination must, therefore, also be in B.

Theorem 3.1 *Assume that for all $f_i, i = 1,2,\cdots,m$, are convex on $\Re^n$, $C \neq \emptyset$, and for some $\hat{x} \in C$ there is a constant $L \equiv L(\hat{x})$ such that $\|\xi\| \leqslant L$ for all $\xi \in \partial f_i^+(x)$ for all $i = 1,2,\cdots,m$ and for all $x \in \Re^n$ for which $\|x - \hat{x}\| \leqslant \|x^0 - \hat{x}\|$ (this assumption will be referred to as "the uniform boundedness of the subgradients"), then the sequence $\{x^k\}$ generated by the algorithm converges to a solution of the convex feasibility problem, i.e., $x^k \longrightarrow x^*$.*

Proof (1) From Proposition 3.1 and Proposition 3.2, we know that the the sequence $\{x^k\}$ generated by the above algorithm is strictly Fejer-monotone with respect to C. That is

$$\|x^{k+1} - x\|^2 \leqslant \|x^k - x\|^2 - \lambda(2-\lambda)\sum_{i\in I_t^k} \omega_i^k \left(\frac{f_i(x^k)}{\|\xi_i^k\|}\right)^2, \tag{3.5}$$

where $x \in C, t = 1,2,\cdots,p$.

(2) We will show that $\lim\limits_{k\to\infty} f_i^+(x^k) = 0$ for all $i = 1,2,\cdots,m$. For $x \in C$ the sequence $\{\|x^k - x\|\}$ is monotonically decreasing, thus

$$\lim_{k\to\infty} \|x^{k+1} - x\| = \lim_{k\to\infty} \|x^k - x\| = d,$$

this implies, from (3.5), for t-th ($t = 1,2,\cdots,p$) subsystem that

$$\lim_{k\to\infty} \sum_{i\in I_t^k} \omega_i^k \left(\frac{f_i(x^k)}{\|\xi_i^k\|}\right)^2 = 0.$$

Since $\omega_{i(t)}^k > \gamma$, for all $i \in I_t$. $t = 1,2,\cdots,p$, then we have

$$\lim_{k\to\infty} f_i^+(x^k)/\|\xi_i^k\| = 0, i = 1,2,\cdots,m.$$

Let $\hat{x}$ be the point whose existence is assumed by the uniform boundedness of the subgradients, and let $C_{\hat{x}} \equiv \{x \in \Re^n | \|x - \hat{x}\| \leqslant \|x^0 - \hat{x}\|\}$. Then $x^k \in C_{\hat{x}}$ for every $k \geqslant 0$, because $\hat{x} \in C, x^k \in \Re^n$ and by repeated application of (3.5). Then $\|\xi_i^k\| \leqslant L$, from which

$$\lim_{k\to\infty} f_i^+(x^k) = 0, \quad i = 1, 2, \cdots, m \tag{3.6}$$

follows.

(3) We will prove that $\lim\limits_{k\to\infty} x^k = x^* \in C$. Since $\{x^k\} \subset C_{\hat{x}}$, which is a compact set, any convergent subsequence of $\{x^k\}$ must satisfy,

$$\lim_{m\to\infty} x^{k_m} = x^*,$$

that is

$$\lim_{m\to\infty} \|x^{k_m} - x^*\| = 0. \tag{3.7}$$

For every $i = 1, 2, \cdots, m$, (3.6) holds for this subsequence, so, by continuity of f_i^+ and $f_i^+(x^*) = 0$, which lead to $x^* \in C$. And from (3.7), we get

$$\lim_{k\to\infty} \|x^k - x^*\| = 0,$$

this completes the proof of the theorem.

Remark 3.1 When the remoste sequence is performed within each block, let $f_{I_t}^j(x^k) = \max\{f_i(x), i \in I_t\}$. Then the parallel iteration within each block become $\omega_{j(t)}^k = 1$ other $\omega_{i(t)}^k = 0, i \neq j$. Therefore, we get the following iteration:

$$x^{k+1} = x^k + \lambda \frac{\max\{f_{I_t}^j(x^k), 0\}}{\|\xi_{I_t^k}\|^2} \xi_{I_t^k}, \tag{3.8}$$

where $\xi_{I_t^k}$ is the subgradient of $f_{I_t}^j(x^k)$ at the point x^k.

The corresponding proof of the convergence is similar to that of the theorem 3.1.

3.2 The parallel block-iterative subgradient projection algorithm and its convergence

In this subsection, we use parallel method not only within block, but also between blocks.

Algorithm 3.2

Initialization: $x^0 \in \Re^n$ is arbitrary.

Iterative step: (i), Given x^k, compute, for all $I = I_1 \bigcup I_2 \bigcup \cdots \bigcup I_p$ the first intermediate iterations $y^{k+1,i}$ are given as

$$y^{k+1,i} = P_{i,\lambda}(x^k) = \begin{cases} x^k - \lambda \dfrac{f_i(x^k)}{\|\xi_i^k\|^2}\xi_i^k, & \text{if } f_i(x^k) > 0; \\ x^k, & \text{if } f_i(x^k) \leqslant 0, \end{cases} \tag{3.9}$$

where $i \in I_t^k, t = 1, 2, \cdots, p,\ \xi_i^k \in \partial f_i(x^k)$ is a subgradient of f_i at the point x^k, and the relaxation parameters λ is confined to an interval $\eta_1 \leqslant \lambda \leqslant 2 - \eta_2$, with $\eta_1, \eta_2 > 0$.

(ii) Compute the second intermediate iteration $\tilde{y}^{k+1,I_t}$by

$$\tilde{y}^{k+1,I_t} = \tilde{P}_{\omega,\lambda}(x^k) = \sum_{i \in I_t^k} \omega_i^k y^{k+1,i} \tag{3.10}$$

where $\omega_i^k, i \in I_t$ are fixed, user-chosen, positive weights with $\sum\limits_{i \in I_t^k} \omega_i^k = 1$.

(iii) Compute the next iteration

$$x^{k+1} = P_{\tau,\omega,\lambda}(x^k) = \sum_{t=1}^{p} \tau_t \tilde{y}^{k+1,I_t}, \tag{3.11}$$

where $\sum\limits_{t=1}^{p} \tau_t = 1$.

Next we discuss the convergence of the parallel block-iterative subgradient projection algorithm.

For the sake of simplicity, we still assume that the weight vector $\omega_t^k = (\omega_i^k : i \in I_t^k)$ is selected so as to satisfy

$$\omega_i^k > \gamma, i \in I_t^k, t = 1, 2, \cdots, p,$$

and $\tau_t, t = 1, \cdots, p$, are assumed to satisfy

$$\tau_t > \upsilon, \quad t = 1, \cdots, p,$$

where γ, υ are some predetermined small positive quantities.

Lemma 3.1[16] *Let p be a positive integer, let $\beta_t \in \Re^n$, for $t = 1, \cdots, p$, and let*

$$\beta = \sum_{t=1}^{p} \tau_t \beta_t,\ \sum_{t=1}^{p} \tau_t = 1, \quad 0 \leqslant \tau_t \leqslant 1,\ t = 1, \cdots, p.$$

Then,

$$\|\beta\|^2 \leqslant \sum_{t=1}^{p} \tau_t \|\beta_t\|^2. \tag{3.12}$$

Proposition 3.3 *Let $x \in C = \bigcap_{i\in I} C_i, \lambda \in [\eta_1, 2-\eta_2]$ with $\eta_1, \eta_2 > 0$ fixed, and let τ_t be weight vector. Then for every $x \in \Re^n$.*

$$\|x^{k+1} - x\| = \|P_{\tau,\omega,\lambda}(x^k) - x\| \leqslant \|x^k - x\|. \tag{3.13}$$

Proof From (3.9)-(3.11), we have

$$x^{k+1} = x^k - \lambda \sum_{t=1}^{p} \tau_t \bigg(\sum_{i\in I_t} \omega_i^k \frac{f_i^k}{\|\xi_i^k\|^2} \xi_i^k \bigg).$$

Let

$$\begin{aligned} \beta_t &= \sum_{i\in I_t} \omega_i^k \frac{f_i^k}{\|\xi_i^k\|^2} \xi_i^k, \\ \beta &= \sum_{t=1}^{p} \tau_t \bigg(\sum_{i\in I_t} \omega_i^k \frac{f_i^k \xi_i^k}{\|\xi_i^k\|^2} \bigg) = \sum_{t=1}^{p} \tau_t \beta_t. \end{aligned}$$

Then

$$\begin{aligned} x^{k+1} - x &= x^k - x - \lambda \sum_{t=1}^{p} \tau_t \bigg(\sum_{i\in I_t} \omega_i^k \frac{f_i^k \xi_i^k}{\|\xi_i^k\|^2} \bigg) \\ &= x^k - x - \lambda\beta. \end{aligned}$$

Furthermore

$$\begin{aligned} \|x^{k+1} - x\|^2 &= \|x^k - x\|^2 + \lambda^2\|\beta\|^2 - 2\lambda\langle \beta, x^k - x\rangle \\ &= \|x^k - x\|^2 + \lambda^2\|\beta\|^2 - 2\lambda \bigg\langle \sum_{t=1}^{p} \tau_t \bigg(\sum_{i\in I_t} \omega_i^k \frac{f_i^k \xi_i^k}{\|\xi_i^k\|^2} \bigg), x^k - x \bigg\rangle \\ &= \|x^k - x\|^2 + \lambda^2\|\beta\|^2 - 2\lambda \sum_{t=1}^{p} \tau_t \bigg(\sum_{i\in I_t} \omega_i^k \frac{f_i^k}{\|\xi_i^k\|^2} \langle \xi_i^k, x^k - x\rangle \bigg). \end{aligned}$$

From the subgradient inequality, it is easy to get

$$\|x^{k+1} - x\|^2 \leqslant \|x^k - x\|^2 + \lambda^2\|\beta\|^2 - 2\lambda \sum_{t=1}^{p} \tau_t \bigg(\sum_{i\in I_t} \omega_i^k \frac{(f_i^k)^2}{\|\xi_i^k\|^2} \bigg).$$

And from (3.12), we have

$$\lambda^2\|\beta\|^2 \leqslant \lambda^2 \sum_{t=1}^{p} \tau_t \|\beta_t\|^2 \leqslant \lambda^2 \sum_{t=1}^{p} \tau_t \bigg(\sum_{i\in I_t} \omega_i^k \frac{(f_i^k)^2}{\|\xi_i^k\|^2} \bigg)$$

and for $0 < \tau_t \leqslant 1, t = 1, \cdots, p;\ 0 < \omega_i^k \leqslant 1, i = 1, \cdots, m$, we obtain

$$\begin{aligned} \|x^{k+1} - x\|^2 &\leqslant \|x^k - x\|^2 - \eta_1\eta_2 \sum_{t=1}^{p} \tau_t \bigg(\sum_{i\in I_t} \omega_i^k \frac{(f_i^k)^2}{\|\xi_i^k\|^2} \bigg) \\ &\leqslant \|x^k - x\|^2. \end{aligned} \tag{3.14}$$

Theorem 3.2 *If for all $i = 1, 2, \cdots, m, f_i$ are convex on $\Re^n$. $C \neq \emptyset$, and for some$\hat{x} \in C$ there is a constant $L \equiv L(\hat{x})$ such that $\|\xi\| \leqslant L$ for all $\xi \in \partial f_i^+(x)$ for all $i = 1, 2, \cdots, m$ and for all $x \in \Re^n$ for which $\|x - \hat{x}\| \leqslant \|x^0 - \hat{x}\|$(this assumption will be referred to as "the uniform boundedness of the subgradients"), then the sequence $\{x^k\}$, produced by the algorithm, converges to a solution of the convex feasibility problem, i.e., $x^k \longrightarrow x^*$.*

Proof (1) From (3.14) we know that the the sequence $\{x^k\}$ generated by the above algorithm is Fejer-monotone with respect to C.

(2) We show that $\lim\limits_{k\to\infty} f_i^+(x^k) = 0$ for all $i = 1, \cdots, m$. For $x \in C$ the sequence $\{\|x^k - x\|\}$ is monotonically decreasing, therefore

$$\lim_{k\to\infty} \|x^{k+1} - c\| = \lim_{k\to\infty} \|x^k - x\| = d.$$

By(3.14), we get

$$\lim_{k\to\infty} \sum_{t=1}^{p} \tau_t \bigg(\sum_{i\in I_t} \omega_i^k \frac{(f_i^k)^2}{\|\xi_i^k\|^2} \bigg) = 0$$

Since

$$\omega_i^k > \gamma,\ \forall i \in I_t,\ \tau_t > \nu, t = 1, 2, \cdots, p,$$

this implies that

$$\lim_{k\to\infty} \frac{(f_i^k)^2}{\|\xi_i^k\|^2} = 0, \quad i = 1, 2, \cdots, m. \tag{3.15}$$

Let $\hat{x}$ be the point whose existence is assumed by the uniform boundedness of the subgradients. Let

$$C_{\hat{x}} \equiv \{x \in \Re^n | \|x - \hat{x}\| \leqslant \|x^0 - \hat{x}\|\}$$

then $x^k \in C_{\hat{x}}$ for every $k \geqslant 0$, because $\hat{x} \in C, x^k \in \Re^n$ and by repeated application of (3.14). Then $\|\xi_i^k\| \leqslant L$, from which

$$\lim_{k\to\infty} f_i^+(x^k) = 0,\ i = 1, 2, \cdots, m \tag{3.16}$$

follows.

(3) We prove that $\lim\limits_{k\to\infty} x^k = x^* \in C$. Since $\{x^k\} \subset C_{\hat{x}}$, which is a compact set, any convergent subsequence of $\{x^k\}$ must satisfy,

$$\lim_{m\to\infty} x^{k_m} = x^*.$$

That is

$$\lim_{m\to\infty} \|x^{k_m} - x^*\| = 0. \tag{3.17}$$

For every $i = 1, \cdots, m$, (3.16) holds for this subsequence, so, by continuity of f_i^+ and $f_i^+(x^*) = 0$, which lead to $x^* \in C$. And from (3.17), we get

$$\lim_{k\to\infty} \|x^k - x^*\| = 0,$$

this completes the proof of the theorem.

Remark 3.2 When the remoste sequence is performed within each block, let $f_{I_t}^j(x^k) = \max\{f_i(x), i \in I_t\}$. Then the parallel iteration within each block become $\omega_{j(t)}^k = 1$ other $\omega_{i(t)}^k = 0, i \neq j$. Therefore, we get the following iteration:

$$x^{k+1} = x^k + \lambda_k \sum_{t=1}^{p} \tau_t \frac{\max\{f_{I_t}^j(x^k), 0\}}{\|\xi_{I_t^k}\|^2} \xi_{I_t^k}. \tag{3.18}$$

where $\xi_{I_t^k}$ is the subgradient of $f_{I_t}^j(x^k)$ at the point x^k.

The corresponding proof of the convergence is similar to that of the theorem 3.2.

4 Some conclusions

In this paper, we establish the sequential block-iterative subgradient projection algorithm and parallel block-iterative subgradient projection algorithm by parting the index set into some subsets. Using subgradient projection can avoid computing the exact projection (sometime which can′t be computed), at the same time using the block-iterative technology can improve the flexibility and the convergence.

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